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XVI. *On the Arithmetic of Impossible Quantities.* By the
 Rev. John Playfair, A. M. Communicated by the
 Rev. Nevil Maskelyne, D. D. F. R. S. and Astronomer
 Royal.

Read Feb. 26, 1778. THE paradoxes which have been introduced into algebra, and remain unknown in geometry, point out a very remarkable difference in the nature of those sciences. The propositions of geometry have never given rise to controversy, nor needed the support of metaphysical discussion. In algebra, on the other hand, the doctrine of negative quantities and its consequences have often perplexed the analyst, and involved him in the most intricate disputation. The cause of this diversity, in sciences which have the same object, must no doubt be sought for in the different modes which they employ to express our ideas. In geometry every magnitude is represented by one of the same kind; lines are represented by a line, and angles by an angle. The genus is always signified by the individual, and a general idea by one of the particulars which fall under it. By this means all contradiction is avoided, and

the geometer is never permitted to reason about the relations of things which do not exist, or cannot be exhibited. In algebra again every magnitude being denoted by an artificial symbol, to which it has no resemblance, is liable, on some occasions, to be neglected, while the symbol may become the sole object of attention. It is not perhaps observed where the connection between them ceases to exist, and the analyst continues to reason about the characters after nothing is left which they can possibly express: if then, in the end, the conclusions which hold only of the characters be transferred to the quantities themselves, obscurity and paradox must of necessity ensue. The truth of these observations will be rendered evident by considering the nature of imaginary expressions, and the different uses to which they have been applied.

2. Those expressions, as is well known, owe their origin to a contradiction taking place in that combination of ideas which they were intended to denote. Thus, if it be required to divide the given line AB (fig. 1.) = a in c , so that $AC \times CB$ may be equal to a given space b^2 , and if $AC=x$, then $x=\frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - b^2}$; which value of x is imaginary when b^2 is greater than $\frac{1}{4}a^2$; now to suppose that b^2 is greater than $\frac{1}{4}a^2$, is to suppose that the rectangle $AC \times CB$ is greater than the square of half the line AB , which

which is impossible. The same holds wherever expressions of this kind occur. Thus, when it is asserted that unity has the three cube roots $1, \frac{-1 + \sqrt{-3}}{2}, \frac{-1 - \sqrt{-3}}{2}$, no more is meant than that when the general equation $x^3 - ax^2 + bx - r = 0$ is, by a change in the data, reduced to the particular state $x^3 - 1 = 0$, x is then equal to unity only, and admits not of any other value, as it does in more general forms of the equation. The natural office of imaginary expressions is, therefore, to point out when the conditions, from which a general formula is derived, become inconsistent with each other; and they correspond in the algebraic calculus to that part of the geometrical analysis, which is usually styled the determination of problems.

3. This, however, is not the only use to which imaginary expressions have been applied. When combined according to certain rules, they have been put to denote real quantities, and though they are in fact no more than marks of impossibility, they have been made the subjects of arithmetical operations; their ratios, their products, and their sums, have been computed, and, what may seem strange, just conclusions have in that way been deduced. Nevertheless, the name of reasoning cannot be given to a process into which no idea is introduced.

Accordingly

Accordingly geometry, which has its modes of reasoning that correspond to every other part of the algebraic calculus, has nothing similar to the method we are now considering; for the arithmetic of mere characters can have no place in a science which is immediately conversant with ideas.

But though geometry rejects this method of investigation, it admits, on many occasions, the conclusions derived from it, and has confirmed them by the most rigorous demonstration. Here then is a paradox which remains to be explained. If the operations of this imaginary arithmetic are unintelligible, why are they not also useless? Is investigation an art so mechanical, that it may be conducted by certain manual operations? or is truth so easily discovered, that intelligence is not necessary to give success to our researches? These are difficulties which it is of some importance to resolve, and on which much attention has not hitherto been bestowed. Two celebrated mathematicians, BERNOULLI and MACLAURIN, have indeed touched on this subject; but being more intent on applying their calculus, than on explaining the grounds of it, they have only suggested a solution of the difficulty, and one too by no means satisfactory. They alledge^(a), that when imaginary expressions are put to

(a) Op. J. BERN. tom. I. N° 70. MACLAUR. Flux. art. 699—763.

denote

denote real quantities, the imaginary characters involved in the different terms of such expressions do then compensate or destroy each other. But beside that, the manner in which this compensation is made, in expressions ever so little complicated, is extremely obscure, if it be considered that an imaginary character is no more than a mark of impossibility, such a compensation becomes altogether unintelligible: for how can we conceive one impossibility removing or destroying another? Is not this to bring impossibility under the predicament of quantity, and to make it a subject of arithmetical computation? And are we not thus brought back to the very difficulty to be removed? Their explanation cannot of consequence be admitted; but, on attempting another, it behoves us to observe, that a more extensive application of this method, than had been made in their time, has now greatly facilitated the inquiry. We begin then with considering the manner in which the imaginary expressions, supposed to denote real quantities, are derived; and the cases in which they prove useful for the purposes of investigation.

4. Let a be an arch of a circle of which the radius is unity, and let c be the number which has unity for its hyperbolic logarithm, then the sine of the arch a , or
fin.

$\sin. \alpha = \frac{c^{\alpha\sqrt{-1}} - c^{-\alpha\sqrt{-1}}}{2\sqrt{-1}}$; and $\cos. \alpha = \frac{c^{\alpha\sqrt{-1}} + c^{-\alpha\sqrt{-1}}}{2}$. These exponential and imaginary values of the sine and cosine are already well known to geometers; and the investigation of them, according to the received arithmetic of impossible quantities, may be as follows.

Let $\sin. \alpha = z$, then $\alpha = \frac{z}{\sqrt{1-z^2}}$. To bring this fluxion under such a form that its fluent may be found by logarithms, both numerator and denominator are to be multiplied by $\sqrt{-1}$; then $\dot{\alpha} = \sqrt{-1} \times \frac{z}{\sqrt{z^2-1}}$, and (by form. 6. HARM. Men.) $\alpha = \sqrt{-1} \times \log. \frac{z+\sqrt{z^2-1}}{\sqrt{-1}}$. Hence $\frac{\alpha}{\sqrt{-1}}$, or $1 \times \frac{\alpha}{\sqrt{-1}} = \log. \frac{z+\sqrt{z^2-1}}{\sqrt{-1}}$, and because 1 is the log. of c , $c^{\frac{\alpha}{\sqrt{-1}}} = \frac{z+\sqrt{z^2-1}}{\sqrt{-1}}$; wherefore, if both parts of the fractional index of c be multiplied by $\sqrt{-1}$, $c^{\alpha\sqrt{-1}} = \frac{z+\sqrt{z^2-1}}{\sqrt{-1}}$. Again, if the arch α be considered as negative, its sine becomes also negative, and therefore $-\alpha = \sqrt{-1} \times \log. \frac{-z+\sqrt{z^2-1}}{\sqrt{-1}}$, or, $-\alpha\sqrt{-1} = -\log. \frac{-z+\sqrt{z^2-1}}{\sqrt{-1}}$, and $\alpha\sqrt{-1} = \log. \frac{-z+\sqrt{z^2-1}}{\sqrt{-1}}$; whence also, $c^{\alpha\sqrt{-1}} = \frac{-z+\sqrt{z^2-1}}{\sqrt{-1}}$. If from this equation the former be taken away, there remains $-\frac{2z}{\sqrt{-1}} = c^{\alpha\sqrt{-1}} - c^{-\alpha\sqrt{-1}}$, whence dividing by $2\sqrt{-1}$ we have $z = \sin. \alpha = \frac{c^{\alpha\sqrt{-1}} - c^{-\alpha\sqrt{-1}}}{2\sqrt{-1}}$. By adding together the

equations a value of the cosine may be found in the same imaginary terms which were assigned above. Now by means of these expressions many theorems may be demonstrated; it may, for example, be shewn, that if a and b are any two arches of a circle, of which the radius is unity, then $\sin. a \times \cos. b = \frac{\sin. a+b}{2} + \frac{\sin. a-b}{2}$. For $\sin. a = \frac{c^{a\sqrt{-1}} - c^{-a\sqrt{-1}}}{2\sqrt{-1}}$, and $\cos. b = \frac{c^{b\sqrt{-1}} + c^{-b\sqrt{-1}}}{2}$, therefore, $\sin. a \times \cos. b = \frac{c^{a+b}\times\sqrt{-1} - c^{-a-b}\times\sqrt{-1} + c^{a-b}\times\sqrt{-1} - c^{-b-a}\times\sqrt{-1}}{4\sqrt{-1}} = \frac{\sin. a+b}{2} + \frac{\sin. a-b}{2}$.

5. Now it may be observed, that the imaginary value which has been found for $\sin. a$ was obtained by bringing a fluxion, properly belonging to the circle, under the form of one belonging to the hyperbola. It may, therefore, be worth while to inquire, whether a similar expression may not be derived from the hyperbola itself.

Let BAD be a rectangular hyperbola (fig. 2.) of which the center is c , and the semi-transverse axis $AC=1$; let B be any point in the hyperbola, join BC , and let BE be an ordinate to the transverse axis. Then, if the sector $ACB = \frac{1}{2}a$, and $BE=z$, it is known that $a=\frac{z}{\sqrt{1+z^2}}$; whence $a=\log. z+\sqrt{1+z^2}$, and $c^a=z+\sqrt{1+z^2}$. But if the sector be taken on the other side of the transverse axis, a and z become

become negative, and $c^{-a} = -z + \sqrt{1+z^2}$. Hence $z = \frac{c-a-c-a}{2}$; in like manner the absciss belonging to ACB, that is $CE = \frac{c+a-c-a}{2}$. These values of the ordinates and abscissæ differ in nothing from those of the sines and cosines already found, except in being free from impossible quantities; for it is evident, that the quantity a is related in the same manner both to the circular and hyperbolic sectors. If now ord. a and abs. b denote the ordinate and absciss belonging to the sectors $\frac{1}{2}a$, $\frac{1}{2}b$ respectively, ord. $a \times$ abs. $b = \frac{ca-c-a}{2} \times \frac{cb+c-b}{2} = \frac{ca+b-c-a-b+ca-b-cb-a}{4} =$
 $\frac{\text{ord. } a+b}{2} + \frac{\text{ord. } a-b}{2}$.

6. The conclusions in both the foregoing cases are perfectly coincident, and the methods by which they have been obtained are similar; though with this difference between them, that in the first all the steps are unintelligible, but in the last significant. If then, notwithstanding a difference which might be expected so materially to affect their conclusions, they have been equally successful in the discovery of truth, it can be ascribed only to the analogy which takes place between the subjects of investigation; an analogy so close, that every property belonging to the one may, with certain restrictions, be transferred to the other. Accordingly,

every imaginary expression, which has been found to belong to the circle in the preceding calculation, is by the substitution of real for impossible quantities, or of $\sqrt{1}$ for $\sqrt{-1}$, converted into a proposition which holds of the hyperbola. The operations, therefore, performed with the imaginary characters, though destitute of meaning themselves, are yet notes of reference to others which are significant. They point out indirectly a method of demonstrating a certain property of the hyperbola, and then leave us to conclude from analogy that the same property belongs also to the circle. All that we are assured of by the imaginary investigation is, that its conclusion may, with all the strictness of mathematical reasoning, be proved of the hyperbola; but if from thence we would transfer that conclusion to the circle, it must be in consequence of the principle which has been just now mentioned. The investigation, therefore, resolves itself ultimately into an argument from analogy; and, after the strictest examination, will be found without any other claim to the evidence of demonstration. Had the foregoing proposition been proved of the hyperbola only, and afterwards concluded to hold of the circle, merely from the affinity of the curves, its certainty would have been precisely the same as when a proof is made out by the intervention of imaginary symbols.

8. Though it might readily be concluded, that the same principle on which the foregoing investigation has been found to proceed, extends itself to all those in which imaginary expressions are put to denote real quantities, it may yet be proper to make trial of its application in some other instances.

Let AB, AC, AD, AE (fig. 3.) be any arches of a circle in arithmetical progression, and let m be their number; it is required to find the sum of the sines $BC, CH, \&c.$ of those arches. Let the radius $AF=1$, $AB=a$, and the common difference of the arches, or $BC=x$: the sum of the series $\sin. a + \sin. a+x + \sin. a+2x + \dots (m)$ is to be found. Now, because $\sin. a = \frac{c^{a\sqrt{-1}} - c^{-a\sqrt{-1}}}{2\sqrt{-1}}$, and $\sin. a+x = \frac{c^{a+x\sqrt{-1}} - c^{-a-x\sqrt{-1}}}{2\sqrt{-1}}$, &c.; the series $\sin. a + \sin. a+x + \sin. a+2x \dots (m) = \frac{c^{a\sqrt{-1}}}{2\sqrt{-1}} \times \frac{1 + c^{x\sqrt{-1}} + c^{2x\sqrt{-1}} \dots (m)}{1 - c^{x\sqrt{-1}} + c^{-2x\sqrt{-1}}} - \frac{c^{a\sqrt{-1}}}{2\sqrt{-1}} \times \frac{1 - c^{-x\sqrt{-1}} + c^{-2x\sqrt{-1}} \dots (m)}{1 + c^{-x\sqrt{-1}} + c^{2x\sqrt{-1}}} \dots (m)$. But these series are both geometrical progressions, and the sum of the first is $\frac{c^{a\sqrt{-1}}}{2\sqrt{-1}} \times \frac{1 - c^{mx\sqrt{-1}}}{1 - c^{x\sqrt{-1}}}$; and of the second, $\frac{c^{-a\sqrt{-1}}}{2\sqrt{-1}} \times \frac{1 - c^{-mx\sqrt{-1}}}{1 - c^{-x\sqrt{-1}}}$.

The sum of the proposed series therefore

$$= \frac{c^{a\sqrt{-1}}}{2\sqrt{-1}} \times \frac{1 - c^{mx\sqrt{-1}}}{1 - c^{x\sqrt{-1}}} - \frac{c^{-a\sqrt{-1}}}{2\sqrt{-1}} \times \frac{1 - c^{-mx\sqrt{-1}}}{1 - c^{-x\sqrt{-1}}} = \frac{1}{2\sqrt{-1}} \times \\ \frac{c^{a\sqrt{-1}} - c^{a+mx\sqrt{-1}} - c^{-a\sqrt{-1}} + c^{-a+mx\sqrt{-1}}}{1 - c^{x\sqrt{-1}} - c^{-x\sqrt{-1}} + 1} + \frac{1}{2\sqrt{-1}} \times \\ \frac{-c^{-a\sqrt{-1}} - c^{a+mx\sqrt{-1}} - c^{-a\sqrt{-1}} + c^{-a+mx\sqrt{-1}}}{1 - c^{x\sqrt{-1}} - c^{-x\sqrt{-1}} + 1} - \frac{1}{2\sqrt{-1}} \times$$

$$\frac{1 - \sin \alpha + \sin \alpha + mx \times \sqrt{-1} + \sin \alpha + mx \times \sqrt{-1} - \sin \alpha + mx \times \sqrt{-1}}{1 - \sin \alpha - \sin \alpha + mx \times \sqrt{-1} + 1};$$

in which expression, if the sines be substituted for their imaginary values, we have

$$\frac{\sin \alpha - \sin \alpha + mx - \sin \alpha - x + \sin \alpha + mx - x}{2 \times 1 - \cot x} -$$

$$\sin \alpha + \sin \alpha + x + \sin \alpha + 2x \dots \dots (m). \quad \text{Q. E. I.}$$

When $AB=BC$, or $\alpha=x$, the proposed series becomes $\sin x + \sin 2x + \sin 3x \dots \dots (m)$, and its value = $\frac{\sin x - \sin m+1 \times x + \sin mx}{2 \times 1 - \cot x}$.

In like manner it will be found, that the sum of the cosines of the same arches, or $\cos \alpha + \cos \alpha + x + \cos \alpha + 2x + \dots \dots (m) = \frac{\cos \alpha - \cos \alpha + mx - \cos \alpha - x + \cos \alpha + mx - x}{2 \times 1 - \cot x}$.

and when $\alpha=x$, $\cos x + \cos 2x + \cos 3x \dots \dots (m) = \frac{\cos mx - \cos m+1 \times x}{2 \times 1 - \cot x} - \frac{1}{2}$.

9. To solve the same problem, in the case of the hyperbola, we must follow the steps which have been traced out by these imaginary operations. Let ABE be an equilateral hyperbola (fig. 4.) of which the center is F, and the transverse axis $AF=1$; let ABF, ACF, ADF, &c. be any factors in arithmetical progression, and let m be their number; it is required to find the sum of all the ordinates BG, CH, DK, &c. belonging to those factors. Let the factor $AFB = \frac{1}{2}\alpha$, and the factor BFC, which is the

common difference of the factors, $= \frac{1}{2}x$: then BG, or ord. $a = \frac{c^a - c^{-a}}{2}$, and CH, or ord. $a+x = \frac{c^{a+x} - c^{a-x}}{2}$; by art.

5. Therefore the series of ordinates, that is,

$$\begin{aligned} BG + CH + DK + \dots (m) &= \frac{c^a}{2} \times \overline{I + c^x + c^{2x} + \dots} (m) - \\ \frac{c^{-a}}{2} \times \overline{I + c^{-x} + c^{-2x} + \dots} (m) &= \frac{c^a}{2} \times \frac{1 - c^{mx}}{1 - c^x} - \frac{c^{-a}}{2} \times \frac{1 - c^{-mx}}{1 - c^{-x}} - \\ = \frac{1}{2} \times \frac{c^a - c^{a+mx} - c^{a-x} + c^{a+mx-x} - c^{-a} + c^{-a-mx} + c^{-a-x} - c^{-a-mx+x}}{1 - c^x - c^{-x} + I} = \\ \text{ord. } a - \text{ord. } a+mx - \text{ord. } a-x + \text{ord. } a+mx-x & \quad \text{When } a=x, \\ 2 \times \overline{I - \text{absl. } x} & \\ \text{ord. } x + \text{ord. } 2x + \text{ord. } 3x + \dots (m) &= \\ \frac{\text{ord. } x - \text{ord. } \overline{m+1} \times x + \text{ord. } mx}{2 \times \overline{I - \text{absl. } x}}. & \end{aligned}$$

In like manner it is proved, that the sum of the abscissæ, that is, FG + FH + FK + (m) = $\frac{\text{absl. } a - \text{absl. } a+mx - \text{absl. } a-x + \text{absl. } a+mx-x}{2 \times \overline{I - \text{absl. } x}}$; and when $a=x$, this expression becomes $\frac{\text{absl. } mx - \text{absl. } \overline{m+1} \times x}{2 \times \overline{I - \text{absl. } x}} - \frac{1}{2}$.

10. The coincidence of the theorems deduced in the two last articles is obvious at first sight, and if the methods by which they have been obtained be compared, it will appear, that the imaginary operations in the one case were of no use but as they adumbrated the real demonstration, which took place in the other. This will be rendered more evident by considering that the resolution of the series of hyperbolic ordinates, into two others

of continual proportionals, can be exhibited geometrically. For, from the points A, B, C, and D, let AM, BN, CO, DP, be drawn at right angles to the asymptote FP; let GB produced meet FP in Q, and let RR be perpendicular to the conjugate axis FR. Then, because the triangles FRS, FMA, are equiangular, $AF : FM :: FS : FR$; hence $FR = \frac{FM}{FA} \times FS = \frac{FM}{FA} \times \overline{FN - NB}$. For the same reason $CH = \frac{FM}{FA} \times \overline{FO - OC}$, and $DK = \frac{FM}{FA} \times \overline{FP - PD}$. Therefore, $BG + CH + DK = \frac{FM}{FA} \times \overline{FN + FQ + FP} - \frac{FM}{FA} \times \overline{BN + CQ + DP}$; now, FN, FO, FP, are continual proportionals, and so also are BN, FO, FP, because the sectors FBC, FCD, are equal. But in the circle no such resolution of the proposed series of sines can take place, that series being subject to alternate increase and diminution; on which account it is, that imaginary characters enter into the exponential value of the sine. Those characters are therefore so far from compensating each other in the present case, as they ought to do, on the supposition of BERNOULLI and MACLAURIN, that they manifestly serve as marks of impossibility. There remains, of consequence, the affinity between circular arches and hyperbolic areas, or between the measures of angles and of ratios, as the only principle on which the imaginary investigation can proceed. It need scarcely be observed,

that

that the exponential value of the hyperbolic ordinate may be deduced from what has been proved in this article.

11. But as the arithmetic of impossible quantities is no where of greater use than in the investigation of fluents, it is of consequence to inquire, whether the preceding theory extends also to that application of it.

Let it then be required to find the fluent of the equation $\frac{y}{z^2} = a^2 y = Q$, where Q denotes any function whatever of x . For this purpose, the following lemma is premised: let x be any arch, and p any flowing quantity; then, if the sign \int , be taken to denote the fluent of the quantity to which it is prefixed, fin. $x \int p \cos. x - \cos. x \int p \sin. x = \frac{c^{x\sqrt{-1}}}{2\sqrt{-1}} \int p c^{-x\sqrt{-1}} - \frac{c^{-x\sqrt{-1}}}{2\sqrt{-1}} \int p c^{x\sqrt{-1}}$; or if $\frac{1}{2}x$ be a hyperbolic sector, ord. $x \int p \text{ abs. } x - \text{abs. } x \int p \text{ ord. } x = \frac{c^x}{2} \int p c^{-x} - \frac{c^{-x}}{2} \int p c^x$.

Because fin. $x \int p \cos. x = \frac{c^{x\sqrt{-1}} - c^{-x\sqrt{-1}}}{2\sqrt{-1}} \int p x \frac{c^{x\sqrt{-1}} + c^{-x\sqrt{-1}}}{2}$, by separating the terms we have fin. $x \int p \cos. x = \frac{c^{x\sqrt{-1}}}{4\sqrt{-1}} \int p c^{x\sqrt{-1}} + \frac{c^{-x\sqrt{-1}}}{4\sqrt{-1}} \int p c^{-x\sqrt{-1}} - \frac{c^{-x\sqrt{-1}}}{4\sqrt{-1}} \int p c^{x\sqrt{-1}} - \frac{c^{x\sqrt{-1}}}{4\sqrt{-1}} \int p c^{-x\sqrt{-1}}$, for the same reason $-\cos. x \int p \sin. x = \frac{c^{x\sqrt{-1}}}{4\sqrt{-1}} \int p c^{x\sqrt{-1}} + \frac{c^{-x\sqrt{-1}}}{4\sqrt{-1}} \int p c^{-x\sqrt{-1}} - \frac{c^{-x\sqrt{-1}}}{4\sqrt{-1}} \int p c^{x\sqrt{-1}} + \frac{c^{x\sqrt{-1}}}{4\sqrt{-1}}$.

$\frac{c^{-x\sqrt{-1}}}{4\sqrt{-1}} \int \dot{p} c^{-x\sqrt{-1}}$. Wherefore, by collecting the sum of all the terms, we have $\sin. x \int \dot{p} \cos. x - \cos. x \int \dot{p} \sin. x = \frac{c^{\sqrt{-1}}}{2\sqrt{-1}} \int \dot{p} c^{-x\sqrt{-1}} - \frac{c^{-x\sqrt{-1}}}{2\sqrt{-1}} \int \dot{p} c^{x\sqrt{-1}}$.

The demonstration in the case of the hyperbola is free from imaginary expressions; but, in other respects, is exactly similar to that which has now been given in the case of the circle.

12. Let the co-efficient of y in the proposed equation be first supposed negative, that is, let $\frac{\dot{y}}{x} - a^2 y = Q$, and if we multiply by $c^{nx}\dot{x}$, n being a constant but indeterminate quantity, it becomes $\frac{c^{nx}\dot{y}}{x} - a^2 c^{nx} y \dot{x} = c^{nx} Q \dot{x}$. Let $c^{nx} \times \frac{A\dot{y}}{x} - B y$ be assumed for the fluent, A and B being indeterminate, and let its fluxion be taken, then,

$$\frac{Ac^{nx}\dot{y}}{x} + n A c^{nx} \dot{y} - n B c^{nx} y \dot{x} = c^{nx} Q \dot{x}.$$

$$- B c^{nx} \dot{y}.$$

Hence, by comparing the terms, we get $A=1$, $nA-B=0$, $nB=a^2$; therefore, $n=\pm a$, and $B=\pm a$: for n and B let the value $+a$ be substituted, and for A , its value, unity; and the assumed equation becomes $\frac{\dot{y}}{x} - a y \times c^{ax} = \int c^{ax} Q \dot{x}$, or $\frac{\dot{y}}{x} - a y = c^{-ax} \int c^{ax} Q \dot{x}$. Let this equation be multiplied by $c^{mx}\dot{x}$, m being indeterminate as before, and

$$c^{mx} \dot{y}$$

$c^{m \times} \dot{y} - a c^{m \times} y \dot{x} = \overline{c^{m-a}} \times \dot{x} \int c^{ax} Q \dot{x}$. The fluent of the first member of this equation is evidently of the form $D c^{m \times} y$, the fluxion of which, viz. $D c^{m \times} \dot{y} + D m c^{m \times} y \dot{x}$ being compared with the former gives $D=1$, and $m=-a$; wherefore, $c^{-ax} y = \int c^{-2ax} \dot{x} \int c^{ax} Q \dot{x}$, or $y = c^{ax} \times \int c^{-2ax} \dot{x} \int c^{ax} Q \dot{x}$. Let $c^{ax} Q \dot{x} = z$, and $c^{-2ax} \dot{x} = v$; then $\int c^{-2ax} \dot{x} \int c^{ax} Q \dot{x} = \int z v = zv - \int v z$; but $v = \frac{1}{2a} - \frac{c^{-2ax}}{2a}$, supposing that v and x vanish at the same time; therefore $zv - \int v z = \frac{1}{2a} \int c^{ax} Q \dot{x} - \frac{c^{-2ax}}{2a} \int c^{ax} Q \dot{x} - \frac{1}{2a} \int c^{ax} Q \dot{x} + \frac{1}{2a} \int c^{-ax} Q \dot{x} = \frac{1}{2a} \int c^{-ax} Q \dot{x} - \frac{c^{-2ax}}{2a} \int c^{ax} Q \dot{x}$. Hence $y = \frac{c^{ax}}{2a} \int c^{-ax} Q \dot{x} - \frac{c^{-ax}}{2a} \int c^{ax} Q \dot{x}$. This value of y is sufficient for the construction of the fluent, because the quantities $\int c^{-ax} Q \dot{x}$, and $\int c^{ax} Q \dot{x}$ depend on the quadrature of the hyperbola; but if we would introduce into it the ordinates and abscissæ of that curve, we need only have recourse to the foregoing lemma, from which it appears, that $y = \frac{1}{a} \text{ ord. } ax \int Q \dot{x} \text{ abs. } ax - \frac{1}{a} \text{ abs. } ax \int Q \dot{x} \text{ ord. } ax$.

13. Let the co-efficient of y be now supposed affirmative, or let $\frac{\ddot{y}}{\dot{x}} + a^2 y = Q$. In this case imaginary expressions are introduced into the fluent, and the construction by

the hyperbola becomes impossible. For we have then, $n = \pm ax\sqrt{-1}$, from which, by proceeding as above, we get
 $y = \frac{c^{ax}\sqrt{-1}}{2ax\sqrt{-1}} \int c^{-ax\sqrt{-1}} Q \dot{x} - \frac{c^{-ax}}{2ax\sqrt{-1}} \int c^{ax\sqrt{-1}} Q \dot{x};$ hence also, by the lemma, $y = \sin. ax \int Q \dot{x} \cos. ax - \cos. ax \int Q \dot{x} \sin. ax$. Here the quantities, $\int Q \dot{x} \cos. ax$, and $\int Q \dot{x} \sin. ax$, are assignable by the quadrature of the circle, in the same manner as $\int Q \dot{x}$ abs. ax , and $\int Q \dot{x}$ ord. ax , by the quadrature of the hyperbola; but the method of investigating them, though an illustration of the principles which we have laid down, is too well known to need to be inserted here. In like manner might the fluents of innumerable fluxionary equations, comprehended under the general form $Q = y + \frac{ay}{x} + \frac{by}{x^2} + \frac{dy}{x^3} + \&c.$ be deduced, and all of them would tend to prove that the arithmetic of impossible quantities is no more than a method of tracing the analogy between the measures of ratios and of angles. M. M. EULER^(b) and D'ALEMBERT^(c) were the first to integrate such equations as the preceding, and the method employed here differs from theirs only by being better adapted to illustrate the principle which is common to them all.

(b) Nov. Com. Petrop. tom. III.

(d) Theorie de la Lune.

14. The forms in the *Harmonia Mensurarum* might also be brought to confirm this theory; but, without accumulating instances any farther, it may be sufficient to remark two consequences that follow from it: 1. That the only cases in which imaginary expressions may be put to denote real quantities, are those in which the measures of ratios or of angles are concerned. 2. That the property of either of those measures, so investigated, might have been inferred from analogy alone. Now both these conclusions are agreeable to experience. It does not appear, that any instance has yet occurred where imaginary characters serve to express real quantities, if circular arches or hyperbolic areas are not the subjects of investigation; and if the conclusion obtained may not be transferred from the one to the other, by a mere substitution of corresponding magnitudes; that is, of sines for ordinates, cosines for abscissæ, and circular arches for the doubles of hyperbolic sectors. The affinity between the circle and hyperbola is not however so close, but that it is subject to certain limitations, from considering which, the truth of what is here asserted will be rendered more evident.

1. Any proposition demonstrated of hyperbolic sectors may be transferred to circular arches by substitution alone, without any change in the signs, when only

abscissæ and their products enter into the enunciation, and conversely. Thus $\text{abs. } a \times \text{abs. } b = \frac{\text{abs. } \overline{a+b}}{2} + \frac{\text{abs. } \overline{a-b}}{2}$; and $\text{cos. } a \times \text{cos. } b = \frac{\text{cos. } \overline{a+b}}{2} + \frac{\text{cos. } \overline{a-b}}{2}$. The same holds when the simple power of the ordinate is combined with any power whatever of the abscissæ: so in the theorems of art. 5. and 4. $\text{ord. } a \times \text{abs. } b = \frac{\text{ord. } \overline{a+b}}{2} + \frac{\text{ord. } \overline{a-b}}{2}$; and $\text{sin. } a \times \text{cos. } b = \frac{\text{sin. } \overline{a+b}}{2} + \frac{\text{sin. } \overline{a-b}}{2}$.

2. When an expression containing any property of hyperbolic sectors, involves in it the rectangle of two ordinates, the value of that rectangle must have a contrary sign, when a transition is made to the circle. Thus $\text{ord. } a \times \text{ord. } b = \frac{\text{abs. } \overline{a+b}}{2} - \frac{\text{abs. } \overline{a-b}}{2}$; but $\text{sin. } a \times \text{sin. } b = -\frac{\text{cos. } \overline{a+b}}{2} + \frac{\text{cos. } \overline{a-b}}{2}$. The difference which according to this rule is found between the powers of ordinates and of sines may be seen in the following examples. If $\frac{1}{2}x$ denote any hyperbolic sector, then; by involving $\frac{x^2 - 1}{2}$, and again substituting for the exponential quantities as in art. 5. we have,

$$\text{ord. } x^2 = \frac{\text{abs. } 2x-1}{2};$$

$$\text{ord. } x^3 = \frac{\text{ord. } 3x-3 \text{ ord. } x}{4};$$

$$\text{ord. } x^4 = \frac{\text{abs. } 4x-4 \text{ abs. } 2x+3}{8};$$

$$\overline{\text{ord. } x}^5 = \frac{\text{ord. } 5x - 5 \text{ ord. } 3x + 10 \text{ ord. } x}{16}; \text{ and universally,}$$

if n be any number; a the co-efficient of the second term of a binomial raised to the power n , b the co-efficient of the third, &c. and p the greatest co-efficient: when n is an even number,

$$\overline{\text{ord. } x}^n = \frac{\text{abs. } nx - a \text{ abs. } n-2 \times x + b \text{ abs. } n-4 \times x \dots \dots \mp p \pm \frac{p}{2^n}}{2^{n-1}};$$

but when n is an odd number,

$$\overline{\text{ord. } x}^n = \frac{\text{ord. } nx - a \text{ ord. } n-2 \times x + b \text{ ord. } n-4 \times x \dots \dots \mp p \text{ ord. } x}{2^{n-1}}.$$

If now x denote an arch of a circle, by substituting and changing the signs as oft as $\overline{\text{ord. } x}^n$ occurs in any of the preceding expressions, we get

$$\overline{\text{fin. } x}^2 = \frac{1 - \text{cof. } 2x}{2};$$

$$\overline{\text{fin. } x}^3 = \frac{3 \text{ fin. } x - \text{fin. } 3x}{4};$$

$$\overline{\text{fin. } x}^4 = \frac{3 - 4 \text{ cof. } 2x + \text{cof. } 4x}{8};$$

$$\overline{\text{fin. } x}^5 = \frac{10 \text{ fin. } x - 5 \text{ fin. } 3x + \text{fin. } 5x}{16}: \text{ and universally, if}$$

n be any number, p the greatest co-efficient of a binomial raised to the power n , A the co-efficient next less than p , B the co-efficient next less than A , and so on: when n is an even number,

$$\overline{\text{fin. } x}^n = \frac{\frac{1}{2}p - A \text{ cof. } 2x + B \text{ cof. } 4x - \&c.}{2^{n-1}};$$

but when n is an odd number,

$$\overline{\text{fin. } x}^n = \frac{\frac{p}{2} \text{ fin. } x - A \text{ cof. } 3x + B \text{ cof. } 5x - \&c.}{2^{n-1}}.$$

These series differ from the former only in the signs, and the arrangement of the terms; and when either n , or $n - 1$, is divisible by 4, the signs remain the same in both.

16. The reason of the foregoing rule for changing the signs is, that the rectangle under two ordinates to the hyperbola is always expressed by the difference of two abscissæ: and that if from the absciss belonging to a greater sector, be subtracted the absciss belonging to a less, the remainder will be affirmative; whereas, if from the cosine of a greater arch be subtracted the cosine of a less, the remainder will be negative. Therefore, that the rectangles, expressed by these remainders, may have the same sign, in both cases, the signs of the remainders must be different.

It appears then, that the second rule, as well as the first, is founded on the principle of analogy when taken with the necessary limitations, and it is likewise evident from the instances which have been produced, that those rules lead to the very same conclusions which are obtained from the imaginary values of the sine and cosine.

There are, however, instances in which the analogy between the circular and hyperbolic areas being wholly interrupted, neither the foregoing rules, nor any of the same kind, can be applied; but this occasions no ambiguity,

guity for the construction required in such cases is by its nature restricted to one of the curves only. Of this kind is the Cotesian theorem, which requires the whole circle to be divided into a given number of equal parts, and therefore cannot be extended to the hyperbola where a similar division is impossible. Others of a like nature may be derived from the general theorems already investigated; for the circle, by returning into itself, often reduces them to a simplicity to which there is nothing analogous in the hyperbola. Many examples of this might be adduced, but the two following may suffice.

i. Let ABCDE (fig. 5.) be a regular polygon inscribed in a circle, and let m be the number of its sides; it is required to find the sum of the lines FA, FB, FC, &c. drawn from any point F in the circumference, to all the angles of the polygon. By the method which in art. 8. was employed to obtain the sum of the sines of a series of arches in arithmetical progression, it will be found, that the sum of the chords of the arches a , $\overline{a+x}$, $\overline{a+2x}$, (m), that is, (making $FA=a$, and $AB=x$) the sum of the chords of the arches FA, FB, FC, &c. = $\frac{\text{cho. } a - \text{cho. } a + mx - \text{cho. } a - x + \text{cho. } a + mx - x}{2 \times 1 - \text{cos. } \frac{1}{2}x}$; but, in the present case, mx is equal to the circumference, and therefore $-\text{cho. } a + mx = +\text{cho. } a$ (the chord of an arch greater than

the circumference being negative); and, for the same reason, $\text{cho. } \overline{a+mx-x} = -\text{cho. } \overline{a-x} = +\text{cho. } \overline{x-a}$. Hence the general expression becomes $\frac{\text{cho. } \overline{a+cho. } \overline{x-a}}{1-\cos. \frac{1}{2}x} = FA + FB + FC + \dots (m)$. If therefore GK be drawn from the center, bisecting the chord AB in H, and meeting the circumference in K, the sum of the chords, that is, $FA + FB + FC + FD + FE = \frac{AF + FB}{FK} \times GK$.

2. Let n be an even number, the rest remaining as above, and let it be required to find the sum of the n powers of the chords, that is, the sum of $\overline{FA}^n + \overline{FB}^n + \overline{FC}^n + \dots (m)$. By reasoning, as in the case of the sines, it will appear that, if p be the greatest co-efficient of a binomial raised to the power n ; A the co-efficient next less than p ; B the co-efficient next less a . and so on, then,

$$\overline{\text{cho. } a}^n = p - 2A \cos. a + 2B \cos. 2a + 2D \cos. 3a + \text{&c.}$$

$$\overline{\text{cho. } a+x}^n = p - 2A \cos. a+x + 2B \cos. 2 \times a+x + 2D \cos. 3 \times a+x, \text{ &c.}$$

$$\overline{\text{cho. } a+2x}^n = p - 2A \cos. a+2x + 2B \cos. 2 \times a+2x + 2D \cos. 3 \times a+2x, \text{ &c.}$$

Each of these vertical columns is to be continued downward, till the number of terms be equal to m , and therefore the sum of the second is mp . The sum of the third, or of $-2A \times \cos. a + \cos. a+x + \cos. a+2x \dots (m)$, by art. 8. is $-2A \times \frac{\cos. a - \cos. a+mx - \cos. a-x + \cos. a+mx-x}{2 \times 1 - \cos. x} =$
(because $mx =$ the circumference)

$-A \times \frac{\text{cof. } a - \text{cof. } a - \text{cof. } \overline{a-x} + \text{cof. } \overline{a-x}}{1 - \text{cof. } x} = 0$. In like manner do the sums of all the subsequent columns vanish; and therefore, cho. $a + \text{cho. } \overline{a+x} + \text{cho. } \overline{a+2x} \dots \dots (m) = mp$. But when n is an even number, $p = \frac{\frac{1}{2}n+1}{\frac{1}{2}n-1} \times \frac{\frac{1}{2}n+2}{\frac{1}{2}n-2} \dots \dots \times \frac{n}{\frac{1}{2}n}$
 $= \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots \dots \frac{n-1}{2}}{1 \cdot 2 \cdot 3 \cdot 4 \dots \dots \frac{1}{2}n} \times 2^{\frac{1}{2}n}$. If therefore the radius be put $= r$, and the expression made homogeneous, we have
 $\overline{FA} + \overline{FB} + \overline{FC} \dots \dots (m) = m \times \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots \dots \frac{n-1}{2}}{1 \cdot 2 \cdot 3 \cdot 4 \dots \dots \frac{1}{2}n} \times 2^{\frac{1}{2}n} r^n$.

Q. E. I.

This last coincides with the forty-first of the curious and difficult propositions published by Dr. STEWART, under the title of general theorems: it is given there without a demonstration, but appears plainly to have been investigated, in a manner altogether rigorous, by that profound geometer. It may therefore be regarded as one of the instances, in which the conclusions of this imaginary arithmetic are verified by the geometrical analysis.

17. The two foregoing propositions being confined to the circle, and yet having been investigated by the help of imaginary expressions, may, at first sight, seem exceptions to the rule, which we have been endeavouring to establish. But it needs only to be remarked, that they are particular cases of certain theorems belonging both

to the circle and hyperbola, and that it was into the investigation of those theorems, that the imaginary expressions were introduced.

The conclusions therefore from the whole are these: that imaginary expressions are never of use in investigation but when the subject is a property common to the measures both of ratios and of angles; that they never lead to any consequence which might not be drawn from the affinity between those measures; and that they are indeed no more than a particular method of tracing that affinity. The deductions into which they enter are thus reduced to an argument from analogy, but the force of them is not diminished on that account. The laws to which this analogy is subject; the cases in which it is perfect, in which it suffers certain alterations, and in which it is wholly interrupted, are capable, as may be concluded from the specimens above, of being precisely ascertained. Supported on so sure a foundation, the arithmetic of impossible quantities will always remain an useful instrument in the discovery of truth, and may be of service when a more rigid analysis can hardly be applied. For this reason, many researches concerning it, which in themselves might be deemed absurd, are nevertheless not destitute of utility. M. BERNOULLI has found, for example, that if r be the radius of a circle, the circumference

Fig. 1.

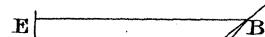
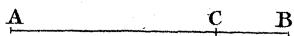


Fig. 3.

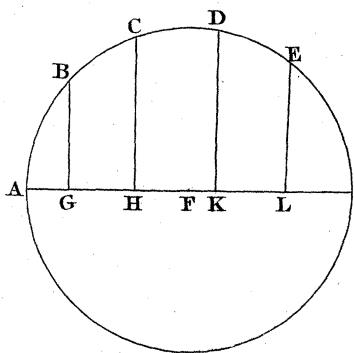


Fig. 2.

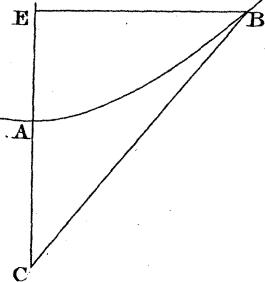


Fig. 4.

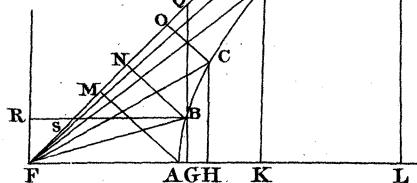
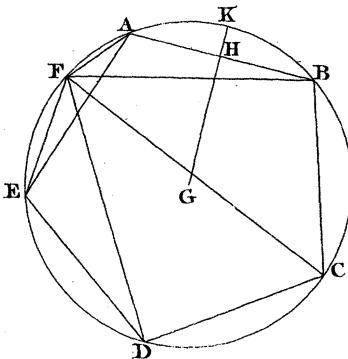


Fig. 5.



$= \frac{4 \log. \sqrt{-1}}{\sqrt{-1}} r$; and the same may be deduced from art. 4. Considered as a quadrature of the circle, this imaginary theorem is wholly insignificant, and would deservedly pass for an abuse of calculation; at the same time we learn from it, that if in any equation the quantity $\frac{\log. \sqrt{-1}}{\sqrt{-1}}$ should occur, it may be made to disappear, by the substitution of a circular arch, and a property, common to both the circle and hyperbola, may be obtained. The same is to be observed of the rules which have been invented for the transformation and reduction of impossible quantities^(e): they facilitate the operations of this imaginary arithmetic, and thereby lead to the knowledge of the most beautiful and extensive analogy which the doctrine of quantity has yet exhibited.

(e) The rules chiefly referred to are those for reducing the impossible roots of an equation to the form $A + B\sqrt{-1}$.

